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The Uniqueness of the 1-System of $Q^{-}(7, q)$, q Odd

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It is known that every m-system of the elliptic polar space $Q^{-}(2n+1, q)$ is an SPG regulus of the ambient space $\mathsf{PG}(2n+1, q)$ of $Q^{-}(2n+1, q)$. From the proof of this result, applied to 1-systems of $Q^{-}(7, q)$, it follows that for q odd, each plane of

1. SOME PROPERTIES OF 1-SYSTEMS OF $Q^{-}(7, q)$

A *1-system* $\mathscr{M}$ of the elliptic quadric $Q^{-}(7, q)$ is a set $\{L_0, L_1, \ldots, L_{q^4}\}$ of q^4+1 lines of $Q^{-}(7, q)$ with the property that every plane of $Q^{-}(7, q)$ containing a line L_i of $\mathscr{M}$ has an empty intersection with $(L_0 \cup L_1 \cup \ldots \cup L_{q^4}) \setminus L_i$. We denote the union of all elements of $\mathscr{M}$ by $\tilde{\mathscr{M}}$. Concerning the generators of $Q^{-}(7, q)$, we have the following general result about m-systems of polar spaces, shown by Shult and Thas in [12].

THEOREM 1.1 (Shult and Thas [12]). *If $\mathscr{M}$ is an m-system of the finite classical polar space P, then for any generator G of P we have*

$$|G \cap \tilde{\mathscr{M}}| = \frac{q^{m+1}-1}{q-1}.$$

From now on, let $\mathscr{M}$ be a 1-system of $Q^{-}(7, q)$ for q odd. In [7] it was mentioned as a corollary that every line of $Q^{-}(7, q)$ meets $\tilde{\mathscr{M}}$ in 0, 1, 2, or $q+1$ points and the latter occurs if and only if the line belongs to $\mathscr{M}$.

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Together with Theorem 1.1, a fruitful observation about the planes of $Q^-(7,q)$ follows. If ζ is a plane of $Q^-(7,q)$, then it has $q+1$ points in common with $\tilde{\mathcal{M}}$ by Theorem 1.1. By the corollary in [7], either no three points of $\zeta \cap \tilde{\mathcal{M}}$ are collinear or $\zeta \cap \tilde{\mathcal{M}}$ consists of the $q+1$ points on a line of $\mathcal{M}$. The famous result of Segre [10, 11], which says that every $(q+1)$-arc of $\mathsf{PG}(2,q)$, q odd, is an irreducible conic, now implies that a plane of $Q^-(7,q)$ either contains a line of $\mathcal{M}$ or meets $\tilde{\mathcal{M}}$ in a conic. This observation will play a crucial role in the remainder of the paper.

In Shult and Thas [12], another result is shown, which is of interest here.

THEOREM 1.2 (Shult and Thas [12]). *If $\mathcal{M}$ is a 1-system of $Q^-(7,q)$ and H is a hyperplane of $\mathsf{PG}(7,q) := \langle Q^-(7,q)\rangle$, then*

(i) *H contains just one line of $\mathcal{M}$ if and only if H is tangent to $Q^-(7,q)$ at a point $x \in \tilde{\mathcal{M}}$,*

(ii) *H contains q^2+1 lines of $\mathcal{M}$ otherwise.*

This theorem has an immediate corollary concerning 5-dimensional subspaces of $\mathsf{PG}(7,q)$.

COROLLARY 1.3. *If β is a 5-dimensional subspace of $\mathsf{PG}(7,q)$ intersecting $Q^-(7,q)$ in a nonsingular elliptic or hyperbolic quadric and β contains q^2+1 lines of $\mathcal{M}$, then $\beta \cap \tilde{\mathcal{M}}$ consists of the points on these q^2+1 lines of $\mathcal{M}$.*

Proof. Suppose that there exists a point $x \in \beta \cap \tilde{\mathcal{M}}$, not on any of the q^2+1 lines of $\mathcal{M}$ in β and let $M \in \mathcal{M}$ be the line of $\mathcal{M}$ through x. Then $\langle \beta, M\rangle$ is a hyperplane of $\mathsf{PG}(7,q)$ which shares at least q^2+2 lines with $\mathcal{M}$, a contradiction to Theorem 1.2. It follows that $\beta \cap \tilde{\mathcal{M}}$ consists exactly of the points on the lines of $\mathcal{M}$ in β. ∎

2. THE 3-DIMENSIONAL HYPERBOLIC QUADRIC, SPANNED BY TWO LINES OF $\mathcal{M}$

Consider two lines $L_1, L_2 \in \mathcal{M}$ and a point $r \in Q^-(7,q)$ such that $\langle L_1, L_2\rangle \subseteq r^\perp$, with $\perp$ the polarity of $Q^-(7,q)$. Then $\langle L_1, L_2\rangle \cap Q^-(7,q)$ is a hyperbolic quadric $Q^+(3,q)$. As each line of $Q^-(7,q)$ not belonging to $\mathcal{M}$ has at most two points in common with $\tilde{\mathcal{M}}$, $\langle L_1, L_2\rangle \cap \tilde{\mathcal{M}}$ consists exactly of the points on L_1 and L_2. Suppose that there exists a plane of $rQ^+(3,q)$, different from $\langle r, L_1\rangle$ and $\langle r, L_2\rangle$, containing a line M of $\mathcal{M}$. Then M meets $Q^+(3,q)$ in a point of $\tilde{\mathcal{M}}$, not on L_1 or L_2, a contradiction.

Consequently, every plane of $rQ^+(3, q)$, except for $\langle r, L_1\rangle$ and $\langle r, L_2\rangle$, meets $\tilde{\mathscr{M}}$ in an irreducible conic.

We introduce the following notation. The two reguli of lines of $Q^+(3, q)$ will be denoted by $\{L_1, L_2, S_1, S_2, ..., S_{q-1}\}$ and $\{N_1, N_2, ..., N_{q+1}\}$. The conics of points of $\tilde{\mathscr{M}}$ in the planes $\langle r, N_i\rangle$ will be denoted by C_i and the ones in the planes $\langle r, S_j\rangle$ by C'_j. The plane $\langle r, L_1\rangle$, respectively $\langle r, L_2\rangle$, contains exactly one point of each C_i, so $\langle r, L_1\rangle \cap \langle r, N_i\rangle$, respectively $\langle r, L_2\rangle \cap \langle r, N_i\rangle$, is tangent to C_i at $L_1 \cap N_i$, respectively $L_2 \cap N_i$. Hence N_i is the polar line of r with respect to C_i. It follows that $\langle r, S_j\rangle \cap \langle r, N_i\rangle$ has either 0 or 2 points in common with C'_j, and so r is an internal point of C'_j.

Lemma 2.1. *There exist two conjugate lines A and $\bar{A}$ over* $\mathsf{GF}(q^2)$*, belonging to the regulus over* $\mathsf{GF}(q^2)$ *containing $\{N_1, N_2, ..., N_{q+1}\}$, such that, for each $j = 1, 2, ..., q-1$, the lines ra_j and $r\bar{a}_j$ are tangent to the conic C'_j at a_j and $\bar{a}_j$, respectively, where $a_j := A \cap S_j$ and $\bar{a}_j := \bar{A} \cap S_j$.*

Proof. The tangents through r of the conic C_1 are the lines $\langle r, N_1\rangle \cap \langle r, L_1\rangle$ and $\langle r, N_1\rangle \cap \langle r, L_2\rangle$, with tangent points $N_1 \cap L_1$ and $N_1 \cap L_2$. Assume that the secants through r of C_1 are $\langle r, N_1\rangle \cap \langle r, S_j\rangle$, for $j = 1, 2, ..., \frac{q-1}{2}$. Consider a conic $C_i \neq C_1$. The lines $\langle r, N_i\rangle \cap \langle r, L_1\rangle$ and $\langle r, N_i\rangle \cap \langle r, L_2\rangle$ are the tangents through r of C_i, with tangent points $N_i \cap L_1$ and $N_i \cap L_2$. Let β be the linear projectivity of $\langle r, L_1, L_2\rangle$ which fixes r and such that the restriction of β to N_i is the linear projectivity from N_i to N_1, defined by the regulus of $Q^+(3, q)$ containing L_1 and L_2. Then C_1 and C_i^β are conics in the plane $\langle r, N_1\rangle$ with the same tangents through r and the same tangent points. On N_1, these tangent points are the points of a hyperbolic quadric $Q^+(1, q)$. In [15], it is shown that $Q^+(1, q)$ defines an equivalence relation on the points of N_1, not on $Q^+(1, q)$, where equivalence is as follows. Consider any irreducible conic C' in $\langle r, N_1\rangle$ such that C' intersects N_1 in the points of $Q^+(1, q)$ and let p be the pole of N_1 with respect to C'. Then two points x, y on $N_1 \setminus Q^+(1, q)$ are equivalent if and only if px and py contain the same number of points of C'. Hence the two equivalence classes correspond to the set of secants of C' through p on the one hand and the set of lines on p missing C' on the other hand. As C_1 and C_i^β intersect N_1 in the same two points and r is the pole of N_1 with respect to C_1 and C_i^β, it follows from the foregoing that these conics either have the same set of secants through r or their sets of secants through r are complementary. This implies that either the secants through r of C_i are given by $\{\langle r, N_i\rangle \cap \langle r, S_j\rangle \mid j = 1, 2, ..., \frac{q-1}{2}\}$ or they are given by $\{\langle r, N_i\rangle \cap \langle r, S_j\rangle \mid j = \frac{q+1}{2}, \frac{q+3}{2}, ..., q-1\}$.

As r is an internal point of each conic C'_j, there are exactly $\frac{q+1}{2}$ secants of $\tilde{\mathscr{M}}$ in each plane $\langle r, S_j\rangle$. We assume that the numbering of the lines N_i is such that for $i = 1, 2, \ldots, \frac{q+1}{2}$, the secants through r of C_i are contained in the planes $\langle r, S_j \rangle$, $j = 1, 2, \ldots, \frac{q-1}{2}$. Hence $|C_i \cap C'_j| = 2$ for $(i, j) \in \{1, 2, \ldots, \frac{q+1}{2}\} \times \{1, 2, \ldots, \frac{q-1}{2}\}$ and for $(i, j) \in \{\frac{q+3}{2}, \frac{q+5}{2}, \ldots, q+1\} \times \{\frac{q+1}{2}, \frac{q+3}{2}, \ldots, q-1\}$. If we put $L_s \cap C_i := a_{si}$, then we obtain on L_s two sets of points $\{a_{si} \mid i = 1, 2, \ldots, \frac{q+1}{2}\}$ and $\{a_{si} \mid i = \frac{q+3}{2}, \frac{q+5}{2}, \ldots, q+1\}$ for $s = 1, 2$. It follows from Fisher [5] that these sets uniquely define points a, $\bar{a}$ on L_1 and b, $\bar{b}$ on L_2, where $ab := A$ and $\overline{ab} := \bar{A}$ belong to the regulus over $\mathsf{GF}(q^2)$ containing $\{N_1, N_2, \ldots, N_{q+1}\}$, such that the planes $\langle r, A \rangle$ and $\langle r, \bar{A} \rangle$ are tangent to all conics $C'_1, C'_2, \ldots, C'_{q-1}$.

Consider in $\mathsf{PG}(4, q) := \langle rQ^+(3, q) \rangle$ the involution θ with center r and axis $\langle L_1, L_2 \rangle$. For each $x \in \langle L_1, L_2 \rangle$, the line rx is fixed by θ, so θ induces an involution on rx with fixed points r and x. It is known (see Hirschfeld [6, p. 140]) that an involution on a line, which has two fixed points r and x, maps each point $y \notin \{r, x\}$ onto the unique point z such that $\{r, x; y, z\} = -1$. Let rx be a secant line of some conic C_i and hence also of some conic C'_j. If $C_i \cap C'_j = \{h_{ij}, k_{ij}\}$, then clearly $\{r, x; h_{ij}, k_{ij}\} = -1$, so $h_{ij}^{\theta} = k_{ij}$. It follows that θ fixes each conic C_i and each conic C'_j. Consequently, the tangent points on the tangents over $\mathsf{GF}(q^2)$ through r of the conics C'_j must be fixed by θ and hence they lie in $\langle L_1, L_2 \rangle$. As these tangent points were already known to lie in the planes $\langle r, A \rangle$ and $\langle r, \bar{A} \rangle$, they must be points on the lines A and $\bar{A}$. ∎

LEMMA 2.2. *The lines A and $\bar{A}$ are independent of the choice of $r \in Q^-(7, q)$ such that $\langle L_1, L_2 \rangle \subseteq r^{\perp}$.*

Proof. Let r and $\tilde{r}$ be two points on $Q^-(7, q)$, with $r, \tilde{r} \in \langle L_1, L_2 \rangle^{\perp}$. The irreducible conics of points of $\tilde{\mathcal{M}}$ in the planes of $\langle \tilde{r}, L_1, L_2 \rangle \cap Q^-(7, q) = \tilde{r}Q^+(3, q)$ will be denoted by $\tilde{C}_i$ and $\tilde{C}'_j$ in such a manner that $\tilde{C}_i = \langle \tilde{r}, N_i \rangle \cap \tilde{\mathcal{M}}$ and $\tilde{C}'_j = \langle \tilde{r}, S_j \rangle \cap \tilde{\mathcal{M}}$, for $i = 1, 2, \ldots, q+1$ and $j = 1, 2, \ldots, q-1$. In $\mathsf{PG}(5, q) := \langle r, \tilde{r}, L_1, L_2 \rangle$, consider the automorphism γ_i which fixes $\langle L_1, L_2 \rangle$ pointwise, maps r onto $\tilde{r}$ and maps C_i onto $\tilde{C}_i$, for $i = 1, 2, \ldots, q+1$. Let C_i and C'_j have two points in common. As $\langle C'_j \rangle^{\gamma_i} = \langle \tilde{C}'_j \rangle$ we have $|\langle \tilde{C}'_j \rangle \cap \tilde{C}_i| = 2$, and so $|\tilde{C}'_j \cap \tilde{C}_i| = 2$. This implies that the irreducible conics of points of $\tilde{\mathcal{M}}$ in $\tilde{r}Q^+(3, q)$ define the same sets of points of size $\frac{q+1}{2}$ on L_1 and L_2 and consequently also the same lines A and $\bar{A}$. This proves the lemma. ∎

3. ON QUADRICS $Q^-(5, q) \subseteq Q^-(7, q)$

In this section, we will show that there exist a lot of elliptic quadrics $Q^-(5, q) \subseteq Q^-(7, q)$ which contain exactly $q^2 + 1$ elements of $\mathcal{M}$.

As in the previous section, we start with two lines $L_1, L_2 \in \mathcal{M}$ and a point r on $Q^-(7, q)$ such that $\langle L_1, L_2 \rangle \subseteq r^\perp$. Consider two lines N_i and S_j such that $|C_i \cap C'_j| = 2$, which is the same as demanding that the line $\langle r, N_i \rangle \cap \langle r, S_j \rangle = rw := W$, with w the common point of N_i and S_j, is a secant of $\tilde{\mathcal{M}}$. Denote the points of $C_i \cap C'_j$ by u_1 and u_2. In the plane $\langle r, N_i \rangle$, the line N_i is the polar line of r with respect to C_i and similarly, S_j is the polar line of r with respect to C'_j in $\langle r, S_j \rangle$. It follows that $\langle L_1, L_2 \rangle^\perp \cap Q^-(7, q)$ consists of the q^2+1 poles of N_i, respectively S_j, with respect to the conics $\tilde{\mathcal{M}} \cap \zeta$, with ζ any generator of $Q^-(7, q)$ through N_i, respectively S_j. Denote the pole of W with respect to C_i by s and the pole of W with respect to C'_j by s', so s is a point on N_i and s' a point on S_j. Now consider the unique lines $M_1, M_2 \in \mathcal{M}$ through u_1, respectively u_2. Then by a foregoing argument, M_1 and M_2 must be contained in $s^\perp$ and $s'^\perp$, and so in $\langle s, s' \rangle^\perp$.

We can repeat the above argument for all points $\tilde{r} \in \langle L_1, L_2 \rangle^\perp \cap Q^-(7, q)$ with the property that $|w\tilde{r} \cap \tilde{\mathcal{M}}| = 2$. Let l_1, l_2 be the points $L_1 \cap N_i$, $L_2 \cap N_i$ and l'_1, l'_2 the points $A \cap S_j$, $\bar{A} \cap S_j$. Then it holds that $\{w, s; l_1, l_2\} = -1$ and $\{w, s'; l'_1, l'_2\} = -1$. As for each point $\tilde{r}$ we work with the same point w, we also have the same lines N_i and S_j and hence the same pairs $\{l_1, l_2\}$ and $\{l'_1, l'_2\}$. So, the points s and s', which are uniquely determined by w, l_1, l_2, respectively by w, l'_1, l'_2, are the same as well for the points r and $\tilde{r}$. Now, the two points of $w\tilde{r} \cap \tilde{\mathcal{M}}$ define two lines $\tilde{M}_1, \tilde{M}_2 \in \mathcal{M}$ which are contained in $\langle s, s' \rangle^\perp$ too. If $\tilde{r}$ varies in $\langle L_1, L_2 \rangle^\perp \cap Q^-(7, q)$, we thus obtain many lines $\tilde{M}_1, \tilde{M}_2$ of $\mathcal{M}$ in $\langle s, s' \rangle^\perp$. The following theorem states that there are q^2+1 such lines of $\mathcal{M}$ in $\langle s, s' \rangle^\perp$.

THEOREM 3.1. *The elliptic quadric $\langle s, s' \rangle^\perp \cap Q^-(7, q) = Q^-(5, q)$ contains exactly q^2+1 elements of $\mathcal{M}$.*

Proof. We shall count how many lines of $\mathcal{M}$ we find in $\langle s, s' \rangle^\perp$ with the above method. First, observe that different choices for $\tilde{r}$ yield disjoint pairs $\{\tilde{M}_1, \tilde{M}_2\}$. For otherwise, some line $M \in \mathcal{M}$ would intersect wr and $w\tilde{r}$ for different points r and $\tilde{r}$ and $\langle w, r, \tilde{r} \rangle$ would be a plane of $Q^-(7, q)$. This is impossible because no two points of $\langle L_1, L_2 \rangle^\perp \cap Q^-(7, q) = Q^-(3, q)$ are collinear.

If there exist α points r in $\langle L_1, L_2 \rangle^\perp \cap Q^-(7, q)$ such that $|wr \cap \tilde{\mathcal{M}}| = 2$, then this yields 2α lines of $\mathcal{M}$ in $\langle s, s' \rangle^\perp$. As there are at most q^2+1 lines of $\mathcal{M}$ in a hyperplane of $\mathsf{PG}(7, q)$, there are certainly at most q^2+1 lines of $\mathcal{M}$ in the 5-dimensional subspace $\langle s, s' \rangle^\perp$, so $2\alpha \leqslant q^2+1$ or $\alpha \leqslant (q^2+1)/2$.

Number the points of $\langle L_1, L_2 \rangle \cap Q^-(7, q) = Q^+(3, q)$, not on L_1 or L_2, as $w_1, w_2, \ldots, w_{q^2-1}$ and let α_i be the number of points r of $\langle L_1, L_2 \rangle^\perp \cap Q^-(7, q)$

such that $|w_i r \cap \tilde{\mathscr{M}}| = 2$. By counting pairs (w_i, r) with $r \in \langle L_1, L_2 \rangle^{\perp} \cap Q^-(7, q)$ and $|w_i r \cap \tilde{\mathscr{M}}| = 2$, we obtain:

$$\sum_{i=1}^{q^2-1} \alpha_i = (q^2+1) \cdot \frac{1}{2}(q^2-1). \tag{1}$$

Suppose that there exists a point w_i with the property that $\alpha_i < (q^2+1)/2$. As $(q^2+1)/2$ is an upper bound for all α_i, this implies that

$$\sum_{i=1}^{q^2-1} \alpha_i < (q^2-1) \cdot \frac{q^2+1}{2},$$

a contradiction to (1). We conclude that $\alpha_i = (q^2+1)/2$ for each w_i and the elliptic quadric $\langle s, s' \rangle^{\perp} \cap Q^-(7, q) = Q^-(5, q)$ contains exactly q^2+1 lines of $\mathscr{M}$. ∎

So, for each $w \in Q^+(3, q)$, not on L_1 or L_2, we have found an elliptic quadric $Q^-(5, q) = \langle s, s' \rangle^{\perp} \cap Q^-(7, q)$ which contains q^2+1 lines of $\mathscr{M}$. Note that there exists a second point w' of $Q^+(3, q)$, not on L_1 or L_2, which yields the same points s and s' and hence the same elliptic quadric, namely the point w' determined by $\langle s, s' \rangle^{\perp} \cap Q^+(3, q) = \{w, w'\}$. Moreover, if one starts with either the point s or the point s', then the elliptic quadric $Q^-(5, q)$ obtained is exactly $\langle w, w' \rangle^{\perp} \cap Q^-(7, q)$. We see that the points of $Q^+(3, q)$ not on either L_1 or L_2 are divided into pairs (w, w') and that each such pair corresponds to a unique other pair (s, s'). We will refer to the set of these pairs as $\mathscr{P}(L_1, L_2)$; clearly $|\mathscr{P}(L_1, L_2)| = (q^2-1)/2$. The set of lines defined by the pairs of $\mathscr{P}(L_1, L_2)$ is denoted by $\mathscr{L}(L_1, L_2)$; that is,

$$\mathscr{L}(L_1, L_2) := \{ww' \mid (w, w') \in \mathscr{P}(L_1, L_2)\}.$$

The set of the quadrics $Q^-(5, q) \subseteq Q^-(7, q)$ obtained as in Theorem 3.1 will be denoted by $\mathscr{Q}$.

For fixed lines L_1, L_2 of $\mathscr{M}$, each quadric $\langle s, s' \rangle^{\perp} \cap Q^-(7, q)$ with $(s, s') \in \mathscr{P}(L_1, L_2)$ contains at least $(q^2+1)/2$ points of $\langle L_1, L_2 \rangle^{\perp} \cap Q^-(7, q) := Q^-(3, q)$ (the corresponding points r), and so contains $Q^-(3, q)$. Moreover, by Corollary 1.3, the 5-dimensional spaces of two such quadrics cannot have a 4-dimensional intersection, so they have exactly the 3-dimensional space of the elliptic quadric $Q^-(3, q)$ in common; that is, these quadrics have exactly $Q^-(3, q)$ in common. Again by Corollary 1.3, the quadric $Q^-(3, q)$ is disjoint from $\tilde{\mathscr{M}}$. We will show that each elliptic quadric $Q^-(5, q)$ through $Q^-(3, q)$ which contains an element of $\mathscr{M}$ belongs to $\mathscr{Q}$.

LEMMA 3.2. *If $B \in \mathscr{M} \setminus \{L_1, L_2\}$ is not contained in any quadric of $\mathscr{Q}$ through $Q^-(3, q)$, then $\langle \gamma, B \rangle$, with $\gamma := \langle L_1, L_2 \rangle^{\perp}$, intersects $Q^-(7, q)$ in a hyperbolic quadric $Q^+(5, q)$.*

Proof. If $B \in \mathscr{M} \setminus \{L_1, L_2\}$ is not a line of any quadric of $\mathscr{Q}$ through $Q^-(3,q)$, then B is disjoint from all these quadrics, by Corollary 1.3. This implies that $\langle \gamma, B \rangle$ intersects each $\langle s, s' \rangle^\perp$, with $(s,s') \in \mathscr{P}(L_1, L_2)$, exactly in γ, or, equivalently, $\langle \gamma, B \rangle^\perp := K$ is disjoint from every line $ss' \in \mathscr{L}(L_1, L_2)$. If $K \cap L_i$ were nonempty for either $i=1$ or $i=2$, then $\langle \gamma, B \rangle$ and $L_i^\perp$ would be contained in a hyperplane of $\mathsf{PG}(7,q)$, so B would have a point in common with the singular quadric $L_i Q^-(3,q)$, a contradiction. Thus K is a line in $\langle L_1, L_2 \rangle$ which is disjoint from $Q^+(3,q) = \langle L_1, L_2 \rangle \cap Q^-(7,q)$, that is, K is an external line of $Q^-(7,q)$. It follows that $\langle \gamma, B \rangle \cap Q^-(7,q)$ is a hyperbolic quadric $Q^+(5,q)$. ∎

4. ON QUADRICS $Q^+(5,q) \subseteq Q^-(7,q)$

We begin this section with some more notation; the one introduced before will still be used. The 3-dimensional subspace $\langle L_1, L_2 \rangle$ will henceforth be denoted by π. In $\pi \cap Q^-(7,q) = Q^+(3,q)$, we have already defined the lines L_1, L_2, A, and $\bar{A}$, as well as the set of pairs $\mathscr{P}(L_1, L_2)$ and the set of lines $\mathscr{L}(L_1, L_2)$. Let T denote the line of π through the points $L_1 \cap A = a$ and $L_2 \cap \bar{A} = \bar{b}$; $\bar{T}$ denotes its conjugate, containing the points $L_1 \cap \bar{A} = \bar{a}$ and $L_2 \cap A = b$. As before, γ is the 3-dimensional subspace $\pi^\perp$, so that $\gamma \cap Q^-(7,q) = Q^-(3,q)$.

LEMMA 4.1. *For each line $ww' \in \mathscr{L}(L_1, L_2)$, $w = N_i \cap S_j$, the lines L_1, L_2 and ww' define a regulus $\mathscr{R}$ of π, consisting of L_1, L_2, and all lines $tt' \in \mathscr{L}(L_1, L_2)$, where t varies on $N_i \setminus \{L_1 \cap N_i, L_2 \cap N_i\}$.*

Proof. The regulus of $Q^+(3,q)$ containing L_1 and L_2 defines a linear projectivity ξ from $N_i = ws$ to $w's'$. Let θ be the involution on $w's'$ with fixed points $L_1 \cap w's'$ and $L_2 \cap w's'$. Then $\xi\theta$ is a linear projectivity from N_i to $w's'$; joining corresponding points of $\xi\theta$ yields a regulus containing L_1, L_2, and all lines $tt' \in \mathscr{L}(L_1, L_2)$ where t varies on $ws \setminus \{L_1 \cap N_i, L_2 \cap N_i\}$. ∎

Remark. Similarly as in the above lemma, one shows that the lines $ww' \in \mathscr{L}(L_1, L_2)$, where w varies on the line $S_j = ws'$, form a regulus which contains A and $\bar{A}$, if considered over $\mathsf{GF}(q^2)$.

LEMMA 4.2. *All $(q^2-1)/2$ lines $ww' \in \mathscr{L}(L_1, L_2)$, as well as L_1 and L_2, belong to the regular spread of π, defined by T and $\bar{T}$.*

Proof. Consider the involution σ in π with axes T and $\bar{T}$. The lines L_1 and L_2 are fixed by σ, because they intersect T and $\bar{T}$. Hence, σ induces on both L_i, $i = 1, 2$, an involution with fixed points $T \cap L_i$ and $\bar{T} \cap L_i$. From

Hirschfeld [6, p. 140] and the fact that $\{T \cap L_i, \bar{T} \cap L_i; L_i \cap ws, L_i \cap w's'\} = -1$, it follows that $(L_i \cap ws)^\sigma = L_i \cap w's'$ for $i = 1, 2$. As a consequence, σ maps ws onto $w's'$. Similarly, A and $\bar{A}$ are also fixed by σ, with fixed points $A \cap T = L_1 \cap T = a$ and $A \cap \bar{T} = L_2 \cap \bar{T} = b$, respectively $\bar{A} \cap T = L_2 \cap T = \bar{b}$ and $\bar{A} \cap \bar{T} = L_1 \cap \bar{T} = \bar{a}$. Here, $\{a, b; ws' \cap A, w's \cap A\} = -1 = \{\bar{a}, \bar{b}; ws' \cap \bar{A}, w's \cap \bar{A}\}$, which implies that $(ws' \cap A)^\sigma = w's \cap A$ and $(ws' \cap \bar{A})^\sigma = w's \cap \bar{A}$. It follows that $(ws')^\sigma = w's$ and we may conclude that $w^\sigma = (ws' \cap ws)^\sigma = w's \cap w's' = w'$ and similarly $s^\sigma = s'$. It is now clear that the lines ww' and ss' are fixed by σ and consequently they intersect T and $\bar{T}$. As this argument holds for all lines ww' and ss' of $\mathscr{L}(L_1, L_2)$, the lemma follows. ∎

We now use the Klein correspondence and consider the lines of π as points on the Klein quadric $\mathscr{K}$. We will use the same notation for lines of π and for the corresponding points on $\mathscr{K}$. The lines of π belonging to the regular spread defined by T and $\bar{T}$ yield points on a 3-dimensional elliptic quadric $\mathscr{E}$ on $\mathscr{K}$, such that $\langle \mathscr{E} \rangle^\perp$, with $\perp$ the polarity of $\mathscr{K}$, is the line $T\bar{T}$. We have seen that the lines $ww' \in \mathscr{L}(L_1, L_2)$ constitute $\frac{q+1}{2}$ reguli containing L_1 and L_2. On $\mathscr{K}$ each such regulus is a circle of $\mathscr{E}$, considered as an inversive plane, through L_1 and L_2. We denote the set of these $\frac{q+1}{2}$ circles by $\mathscr{C}$. Note that $(L_1 L_2)^\perp = A\bar{A}$ with respect to $\mathscr{E}$, where $A\bar{A}$ is an external line of $\mathscr{E}$. Finally, the regulus consisting of the lines $ww' \in \mathscr{L}(L_1, L_2)$, where w varies on ws', translates to a circle C of $\mathscr{E}$ in a plane through the line $A\bar{A}$.

It is known that for q odd, an equivalence relation can be defined on the set of circles of $\mathscr{E}$, or, equivalently, on the points of $\mathsf{PG}(3, q) = \langle \mathscr{E} \rangle$ not on $\mathscr{E}$, in the following way. Embed $\mathsf{PG}(3, q)$ as a hyperplane in $\mathsf{PG}(4, q)$ and consider in $\mathsf{PG}(4, q)$ a parabolic quadric $Q(4, q)$ such that $\mathsf{PG}(3, q) \cap Q(4, q) = \mathscr{E}$. If the point p is the pole of $\mathsf{PG}(3, q)$ with respect to the polarity of $Q(4, q)$, then equivalence is defined as follows. Two points x, y of $\mathsf{PG}(3, q) \setminus \mathscr{E}$ are equivalent if and only if px and py meet $Q(4, q)$ in the same number of points. Hence we obtain two classes: the class consisting of the points x such that px is disjoint from $Q(4, q)$ and the class of the points y such that py is a secant of $Q(4, q)$. This defines an equivalence relation on the circles of $\mathscr{E}$, where two circles are equivalent if and only if their poles with respect to $\mathscr{E}$ are equivalent. For more details on this topic, including an algebraic approach, we refer to [2, Sect. 7; 3, Sect. 6.4.3; 4; 8, p. 24; 14, p. 84; 15]. The following two lemmas are important to us.

LEMMA 4.3 (Fisher and Thas [4]). *Let $\mathscr{F}$ be a flock of $\mathscr{E}$. If N and S are the carriers of $\mathscr{F}$ and K is a circle of $\mathscr{F}$, then the circles of the pencil intersecting in N and S that contain a point of K are all in the same equivalence class.*

LEMMA 4.4 (Fisher and Thas [4]). *Let $\mathcal{F}$ be the linear flock of $\mathcal{E}$ with carriers N and S. If K is any circle for which all the circles of the pencil intersecting in N and S that contain a point of K are in the same class, then K belongs to $\mathcal{F}$.*

These two lemmas will be used to prove the following one.

LEMMA 4.5. *If B is a line of $\mathcal{M}\setminus\{L_1, L_2\}$ such that $\langle\gamma, B\rangle \cap Q^-(7, q)$ is hyperbolic, then either $B' := \langle\gamma, B\rangle \cap \pi$ is a line of the regular spread defined by T and $\bar{T}$ or B' meets both A and $\bar{A}$.*

Proof. Consider the line B' as a point on the Klein quadric. If B' does not belong to the regular spread defined by T and $\bar{T}$, then the point B' is not a point of $\mathcal{E}$. It is easily seen that $B'^{\perp} \cap \langle\mathcal{E}\rangle$ is a plane intersecting $\mathcal{E}$ in a circle C'. This circle C' is disjoint from the circles of $\mathcal{C}$, for otherwise the line B' would have a point in common with some line $ww' \in \mathcal{L}(L_1, L_2)$. This would imply that $\langle\gamma, B\rangle$ and $\langle s, s'\rangle^{\perp}$, where (s, s') is the pair corresponding to (w, w'), lie in a hyperplane of $\mathsf{PG}(7, q)$ and B would intersect $\langle s, s'\rangle^{\perp} \cap Q^-(7, q)$ in a point, a contradiction to Corollary 1.3.

From Lemma 4.3, applied to the linear flock of $\mathcal{E}$ consisting of the circles in the planes through $A\bar{A}$, it follows that the $\frac{q+1}{2}$ circles of $\mathcal{C}$ belong to the same equivalence class, because they all meet the circle C as we have seen in the remark following Lemma 4.2. Hence the $\frac{q+1}{2}$ other circles on L_1, L_2 also belong to the same equivalence class. As C' is disjoint from the $\frac{q+1}{2}$ circles of $\mathcal{C}$ through L_1, L_2, the other $\frac{q+1}{2}$ circles of the pencil through L_1, L_2, which all belong to the same equivalence class, must all have two points in common with C'. From Lemma 4.4, it now follows that C' is a circle of the linear flock of $\mathcal{E}$ defined by $A\bar{A}$, which means that the line B' of π meets both A and $\bar{A}$. ∎

Note that if B' meets A and $\bar{A}$, it is a line of the regular spread defined by A and $\bar{A}$.

THEOREM 4.6. *For every line $L \in \mathcal{M}$ such that $\langle\gamma, L\rangle \cap Q^-(7, q)$ is nonsingular, the quadric $\langle\gamma, L\rangle \cap Q^-(7, q)$ contains q^2+1 lines of $\mathcal{M}$.*

Proof. If $\langle\gamma, L\rangle \cap Q^-(7, q)$ is an elliptic quadric $Q^-(5, q)$, this result has been shown in Theorem 3.1 and Lemma 3.2.

So we focus on the hyperbolic quadrics $\langle\gamma, B\rangle \cap Q^-(7, q)$, $B \in \mathcal{M}\setminus\{L_1, L_2\}$, and for such quadrics we consider the line $B' := \langle\gamma, B\rangle \cap \pi$. Suppose that two distinct hyperbolic quadrics $\langle\gamma, B_1\rangle \cap Q^-(7, q) := Q_1^+(5, q)$ and $\langle\gamma, B_2\rangle \cap Q^-(7, q) := Q_2^+(5, q)$ are contained in some hyperplane H, with $H \cap Q^-(7, q) = Q(6, q)$. Then the lines $B_1' = \langle\gamma, B_1\rangle \cap \pi$ and $B_2' = \langle\gamma, B_2\rangle \cap \pi$ have a point p in common. It is impossible that H contains a third 5-dimensional subspace $\langle\gamma, B_3\rangle$, $B_3 \in \mathcal{M}$, because in that case

B'_1, B'_2 and B'_3 would lie in a plane of π; so two lines B'_i and B'_j, belonging to the same regular spread of π, would have a point in common, a contradiction. Consequently, all q^2+1 lines of $Q(6, q)$ in $\mathscr{M}$ are contained either in $Q_1^+(5, q)$ or in $Q_2^+(5, q)$.

For each line $M \in \mathscr{M} \setminus \{L_1, L_2\}$, which is not contained in $Q(6, q)$, the plane $\langle H^\perp, M \rangle$ intersects $Q^-(7, q)$ in two lines, M and some other line K. In [7], it is shown that such planes $\langle H^\perp, M \rangle$ contain $q+2$ points of $\tilde{\mathscr{M}}$: the $q+1$ points on M and one other point m' on K. If $M' \in \mathscr{M}$ is the line of $\mathscr{M}$ on which m' lies, then $H^\perp$ is a point of $\langle M, M' \rangle$ and hence $\langle M, M' \rangle^\perp \subseteq H$. This means that H contains a second 3-dimensional subspace γ' playing the same role as γ and we can consider the 5-dimensional spaces $\langle \gamma', B_1 \rangle$ and $\langle \gamma', B_2 \rangle$. If one of them intersects $Q^-(7, q)$ in a $Q^-(5, q)$, then, by Theorem 3.1 and Lemma 3.2, this space contains q^2+1 elements of $\mathscr{M}$, so $\langle \gamma', B_1 \rangle = \langle \gamma', B_2 \rangle$. As at least one of $\langle \gamma, B_1 \rangle$ or $\langle \gamma, B_2 \rangle$ is generated by its lines in $\mathscr{M}$, this space, say $\langle \gamma, B_1 \rangle$, coincides with $\langle \gamma', B_1 \rangle = \langle \gamma', B_2 \rangle$, and so $\langle \gamma, B_1 \rangle \cap Q^-(7, q)$ is elliptic, a contradiction. So $\langle \gamma', B_i \rangle \cap Q^-(7, q) = Q_i^+(5, q)'$ for $i = 1, 2$. Remark that all lines of $\mathscr{M}$ in $Q(6, q)$ are contained in both $Q_1^+(5, q) \cup Q_2^+(5, q)$ and $Q_1^+(5, q)' \cup Q_2^+(5, q)'$. If $Q_1^+(5, q)' = Q_2^+(5, q)'$, then $\langle \gamma', B_1 \rangle = \langle \gamma', B_2 \rangle$ and this space contains all q^2+1 lines of $\mathscr{M}$ in $Q(6, q)$. As above, $\langle \gamma, B_1 \rangle$ or $\langle \gamma, B_2 \rangle$ must hence coincide with $\langle \gamma', B_1 \rangle = \langle \gamma', B_2 \rangle$, which implies that B_1, B_2, and γ are in the same 5-dimensional space, a contradiction to our assumption. We may thus assume that $\langle \gamma', B_1 \rangle \neq \langle \gamma', B_2 \rangle$.

If it is possible to choose γ' such that it is not contained in $\langle \gamma, B_1 \rangle$ nor in $\langle \gamma, B_2 \rangle$, then all four quadrics $Q_i^+(5, q)$, $Q_i^+(5, q)'$, $i = 1, 2$, are distinct and so $Q_i^+(5, q) \cap Q_j^+(5, q)'$, $i, j \in \{1, 2\}$ contains at most two lines of $\mathscr{M}$. It follows that $q^2+1 \leqslant 8$, so $q < 3$, a contradiction.

So suppose that all subspaces γ' are either contained in $\langle \gamma, B_1 \rangle$ or in $\langle \gamma, B_2 \rangle$. It follows from Theorem 1.1 and a simple counting argument that each hyperbolic quadric $Q^+(5, q)$ on $Q^-(7, q)$ contains

$$\frac{2(q+1)(q^2+1)\cdot(q+1)}{2(q+1)} = (q+1)(q^2+1)$$

points of $\tilde{\mathscr{M}}$. As there are lines of $\mathscr{M}$ in both $\langle \gamma, B_1 \rangle$ and $\langle \gamma, B_2 \rangle$ by assumption, this implies that there are also points of $\tilde{\mathscr{M}}$ in $\langle \gamma, B_i \rangle$, $i = 1, 2$, the line of $\mathscr{M}$ through which is not contained in $\langle \gamma, B_i \rangle$. Moreover, each line of $\mathscr{M}$ in $\langle \gamma, B_i \rangle$ forces the existence of $q+1$ points of $\tilde{\mathscr{M}}$ in $\langle \gamma, B_j \rangle$, not on a line of $\mathscr{M}$ in $\langle \gamma, B_j \rangle$, $j \neq i$. So there exist points $x \in \tilde{\mathscr{M}}$, contained in $\langle \gamma, B_i \rangle$, $i = 1, 2$, such that the line of $\mathscr{M}$ through x is not contained in H. Suppose that $x \in \tilde{\mathscr{M}}$ is such a point in $\langle \gamma, B_1 \rangle$. Then, using the same argument as above, we can find a 3-dimensional subspace γ_x ($\neq \gamma$) playing the same role as γ, which is contained in $x^\perp$. Now $x^\perp \cap Q_1^+(5, q)$ is a singular quadric $xQ_x^+(3, q)$. As $\gamma_x \cap Q^-(7, q)$ is an elliptic quadric $Q_x^-(3, q)$, γ_x cannot be contained in $\langle \gamma, B_1 \rangle$, for there are no 3-dimensional subspaces in $\langle \gamma, B_1 \rangle \cap x^\perp$ intersecting $xQ_x^+(3, q)$ in an elliptic quadric $Q^-(3, q)$. We have thus shown that it is possible to find a subspace γ' ($\neq \gamma$) in $\langle \gamma, B_1 \rangle$ and another subspace γ'' ($\neq \gamma$) in $\langle \gamma, B_2 \rangle$, both playing the same role as γ. As $\langle \gamma', B_1 \rangle = \langle \gamma, B_1 \rangle$ and the lines of $\mathscr{M}$ in $Q(6, q)$ are contained in $\langle \gamma, B_1 \rangle \cup \langle \gamma, B_2 \rangle$ on the one hand and in $\langle \gamma', B_1 \rangle \cup \langle \gamma', B_2 \rangle$ on the other hand, there are at most two lines of $\mathscr{M}$ in $\langle \gamma, B_2 \rangle$ and in $\langle \gamma', B_2 \rangle$, namely the ones in their intersection. Similarly there are at most two lines of $\mathscr{M}$ in $\langle \gamma, B_1 \rangle$ and in $\langle \gamma'', B_1 \rangle$, again the two possible lines in their intersection. This implies that $q^2 + 1 \leqslant 4$, so $q < 2$, a contradiction.

It follows from the above that no two distinct quadrics $\langle \gamma, B_1 \rangle \cap Q^-(7, q)$ and $\langle \gamma, B_2 \rangle \cap Q^-(7, q)$, $B_1, B_2 \in \mathscr{M} \setminus \{L_1, L_2\}$, are contained in a hyperplane. Hence for every $Q(6, q) \subseteq Q^-(7, q)$ containing $\langle \gamma, B \rangle \cap Q^-(7, q)$ for some line $B \in \mathscr{M} \setminus \{L_1, L_2\}$, the $q^2 + 1$ lines of $\mathscr{M}$ in $Q(6, q)$ are contained in $\langle \gamma, B \rangle$. ∎

Remark. Let L_1, L_2, L_3 be three lines of $\mathscr{M}$. Then $\langle L_1, L_2, L_3 \rangle$ is 5-dimensional and there exists a nonsingular $Q(6, q) \subseteq Q^-(7, q)$ such that $\langle L_1, L_2, L_3 \rangle \subseteq \langle Q(6, q) \rangle$. Similarly as in the proof of Theorem 4.6, we can find a subspace $\gamma = \langle M, M' \rangle^\perp$, for some $M, M' \in \mathscr{M}$, which is contained in $\langle Q(6, q) \rangle$. From Theorem 4.6, it now follows that the $q^2 + 1$ lines of $\mathscr{M}$ in $Q(6, q)$ are contained in $\langle \gamma, L_1 \rangle = \langle L_1, L_2, L_3 \rangle$. Hence the 5-dimensional subspace spanned by any three elements of $\mathscr{M}$ contains exactly $q^2 + 1$ lines of $\mathscr{M}$. It also follows that $\langle L_1, L_2, L_3 \rangle \cap Q^-(7, q)$ is nonsingular.

If $L_1, L_2, L_3 \in \mathscr{M}$ are such that $\langle L_1, L_2, L_3 \rangle \cap Q^-(7, q)$ is a hyperbolic quadric $Q^+(5, q)$, then the $q^2 + 1$ lines of $\mathscr{M}$ in $Q^+(5, q)$ constitute a 1-system $\mathscr{M}'$ of $Q^+(5, q)$. In Shult and Thas [12], it is shown that the 1-system of $Q^+(5, q)$, q odd, is unique up to a projectivity. It is constructed as follows. There exist two disjoint and conjugate planes π and $\bar{\pi}$ in the extension $\mathsf{PG}(5, q^2)$ of $\mathsf{PG}(5, q) := \langle Q^+(5, q) \rangle$, which are polar with respect to the polarity of $Q^+(5, q^2)$ and such that π, respectively $\bar{\pi}$, intersects $Q^+(5, q^2)$ in an irreducible conic C, respectively $\bar{C}$. The lines of $\mathscr{M}'$ are then all lines $x\bar{x}$, where x varies on C and $\bar{x} \in \bar{C}$ is the conjugate of x.

LEMMA 4.7. *If $\mathscr{M}'$ is a 1-system of $Q^+(5, q)$, q odd, then the planes π and $\bar{\pi}$ are the unique planes in the extension $\mathsf{PG}(5, q^2)$ of $\mathsf{PG}(5, q)$ which meet all lines of $\mathscr{M}'$.*

Proof. Consider $Q^+(5, q)$ as the Klein quadric of the lines of $\mathsf{PG}(3, q)$. With the $q^2 + 1$ lines of $\mathscr{M}$ there correspond $q^2 + 1$ line pencils in $\mathsf{PG}(3, q)$, with respective vertices $p_0, p_1, ..., p_{q^2}$ and contained in the respective planes $\alpha_0, \alpha_1, ..., \alpha_{q^2}$. As $\mathscr{M}'$ is a 1-system, the set of points $\{p_0, p_1, ..., p_{q^2}\}$ is an ovoid of $\mathsf{PG}(3, q)$, which is an elliptic quadric $Q^-(3, q)$ by Barlotti [1] and Panella [9]; the planes $\alpha_0, \alpha_1, \ldots, \alpha_{q^2}$ are the tangent planes of $Q^-(3, q)$. We now consider the extension $Q^+(5, q^2)$ of $Q^+(5, q)$ and the corresponding extension $\mathsf{PG}(3, q^2)$ of $\mathsf{PG}(3, q)$. The conics C and $\bar{C}$, being contained in polar planes with respect to $Q^+(5, q^2)$, correspond to the two reguli of a hyperbolic quadric $\mathscr{H}$ in $\mathsf{PG}(3, q^2)$. As all lines of $\mathscr{M}'$ meet C and $\bar{C}$ in a point, every line of $\mathscr{H}$ contains a unique point of $Q^-(3, q)$, so $Q^-(3, q)$ is contained in $\mathscr{H}$.

Suppose that there exists a third plane β in $\mathsf{PG}(5, q^2)$, meeting all lines of $\mathscr{M}'$. If β is a plane of $Q^+(5, q^2)$, then it either corresponds to the set of lines in a plane of $\mathsf{PG}(3, q^2)$ or to the set of lines through a point of $\mathsf{PG}(3, q^2)$. In the first case, all points of $Q^-(3, q)$ must lie in a plane, while in the second case the q^2+1 tangent planes of $Q^-(3, q)$ must have a common point. In either case, a contradiction arises. Hence $\beta \cap Q^+(5, q^2)$ is a conic C'; let C'' denote the conic $C'^{\perp} \cap Q^+(5, q^2)$. Then C' and C'' yield a hyperbolic quadric $\mathscr{H}'$ in $\mathsf{PG}(3, q^2)$, also containing $Q^-(3, q)$. As $\mathscr{H}$ and $\mathscr{H}'$ have common tangent planes at the points of $Q^-(3, q)$, the points of $Q^-(3, q)$ are contained in a plane, a contradiction. It follows that the planes π and $\bar{\pi}$ are indeed the unique planes meeting all lines of $\mathscr{M}'$. ∎

5. THE UNIQUENESS OF THE 1-SYSTEM OF $Q^-(7, q)$, q ODD

To prove the desired uniqueness result for 1-systems of $Q^-(7, q)$, q odd, we will rely on the theory of eggs. An *egg* of $\mathsf{PG}(2n+m-1, q)$ is a set $O(n, m, q)$ of q^m+1 $(n-1)$-dimensional subspaces $\pi_0, \pi_1, \ldots, \pi_{q^m}$ of $\mathsf{PG}(2n+m-1, q)$, every three of which generate a $\mathsf{PG}(3n-1, q)$ and such that each element π_i of $O(n, m, q)$ is contained in a $\mathsf{PG}(n+m-1, q)$, having no point in common with $(\pi_0 \cup \pi_1 \cup \ldots \cup \pi_{q^m}) \setminus \pi_i$. The space $\mathsf{PG}(n+m-1, q)$ is called the *tangent space* of $O(n, m, q)$ at π_i. An egg $O(n, m, q)$ is called *good* at an element π_i if for all distinct π_j, π_k, $j \neq i \neq k$, the space generated by π_i, π_j, and π_k contains exactly $q^{m-n}+1$ elements of $O(n, m, q)$. If $m = 2n$, an egg $O(n, 2n, q)$ is called *regular* if it is constructed in the following way.

Consider the algebraic extension $\mathsf{GF}(q^n)$ of $\mathsf{GF}(q)$ and the corresponding extension $\mathsf{PG}(4n-1, q^n)$ of $\mathsf{PG}(4n-1, q)$. Consider n 3-dimensional subspaces $\mathsf{PG}^{(1)}(3, q^n), \mathsf{PG}^{(2)}(3, q^n), \ldots, \mathsf{PG}^{(n)}(3, q^n)$ of $\mathsf{PG}(4n-1, q^n)$, which generate the space $\mathsf{PG}(4n-1, q^n)$ and constitute a conjugate n-tuple with respect to the extension $\mathsf{GF}(q^n)$ of $\mathsf{GF}(q)$. Let $\mathscr{O}$ be an ovoid of $\mathsf{PG}^{(1)}(3, q^n)$. With every point $p^{(1)}$ of $\mathscr{O}$, there correspond $n-1$ points $p^{(2)}, p^{(3)}, \ldots, p^{(n)}$ such that the points $p^{(1)}, p^{(2)}, \ldots, p^{(n)}$ constitute a conjugate n-tuple with respect to the extension $\mathsf{GF}(q^n)$ of $\mathsf{GF}(q)$. The points $p^{(1)}, p^{(2)}, \ldots, p^{(n)}$ define an $(n-1)$-dimensional subspace of $\mathsf{PG}(4n-1, q)$. If we let $p^{(1)}$ vary in $\mathcal{O}$, we obtain $q^{2n}+1$ such $(n-1)$-dimensional subspaces of $\mathsf{PG}(4n-1, q)$, which form a regular egg $O(n, 2n, q)$. Regular eggs are good at each of their elements. In [13], Thas shows that the converse also holds.

THEOREM 5.1 (Thas [13]). *Every egg of* $\mathsf{PG}(4n-1, q)$ *which is good at each of its elements is regular.*

We will use this theorem to prove the uniqueness of the 1-system of $Q^-(7, q)$, q odd. If q is odd, the ovoid $\mathcal{O}$ of $\mathsf{PG}^{(1)}(3, q^n)$ is an elliptic quadric by Barlotti [1] and Panella [9]. In such a case the corresponding egg of $\mathsf{PG}(4n-1, q)$ is called *classical.*

THEOREM 5.2. *For q odd, the quadric* $Q^-(7, q)$ *has a unique 1-system up to a projectivity.*

Proof. It is easily seen that each 1-system $\mathcal{M}$ of $Q^-(7, q)$ is an egg $O(2, 4, q)$ of $\mathsf{PG}(7, q)$, where the tangent space of the egg at a line $L \in \mathcal{M}$ is simply the tangent space at L of $Q^-(7, q)$. As we have seen in the remark following Theorem 4.6, $\mathcal{M}$, considered as an egg $O(2, 4, q)$, is good at each of its elements. So, by Theorem 5.1, every 1-system of $Q^-(7, q)$ is a regular egg and as q is odd, this egg is classical. Hence there exist two disjoint, conjugate 3-dimensional subspaces ρ and $\bar{\rho}$ in the extension $\mathsf{PG}(7, q^2)$ of $\mathsf{PG}(7, q)$, such that the lines of $\mathcal{M}$ are all lines $x\bar{x}$, where x is a point of a classical ovoid $\mathcal{O} = Q^-(3, q^2)$ in ρ and $\bar{x}$ is its conjugate, belonging to the conjugate ovoid $\bar{\mathcal{O}}$ in $\bar{\rho}$. Since the lines of $\mathcal{M}$ are lines of $Q^-(7, q)$, their extensions are lines of the extension $Q^+(7, q^2)$ of $Q^-(7, q)$, which implies that $\mathcal{O} \subseteq \rho \cap Q^+(7, q^2)$ and $\bar{\mathcal{O}} \subseteq \bar{\rho} \cap Q^+(7, q^2)$. It remains to show that ρ and $\bar{\rho}$ are polar with respect to the polarity of $Q^+(7, q^2)$.

Consider two arbitrary lines $L_1, L_2 \in \mathcal{M}$. We investigate how many hyperbolic quadrics $Q^+(5, q)$ there are which contain L_1, L_2 and q^2-1 other lines of $\mathcal{M}$. Let σ denote a 5-dimensional subspace through $\langle L_1, L_2\rangle$; then $\langle L_1, L_2\rangle^\perp := \gamma$ contains the line $\sigma^\perp$. Suppose that σ_1 and σ_2 intersect $Q^-(7, q)$ in elliptic quadrics $Q_1^-(5, q)$ and $Q_2^-(5, q)$ respectively, both containing q^2+1 lines of $\mathcal{M}$. In that case $\sigma_1^\perp$ and $\sigma_2^\perp$ are secants of $\gamma \cap Q^-(7, q) := Q^-(3, q)$. If $\sigma_1^\perp$ and $\sigma_2^\perp$ would have a common point p, the hyperplane $p^\perp$ would contain both $Q_1^-(5, q)$ and $Q_2^-(5, q)$, which means that there would be $2q^2$ lines of $\mathcal{M}$ in $p^\perp$, a contradiction to Theorem 1.2. Hence $\sigma_1^\perp$ and $\sigma_2^\perp$ are disjoint and so there are at most $\frac{q^2+1}{2}$, the number of disjoint pairs of points of $Q^-(3, q)$, elliptic quadrics $Q^-(5, q)$ through L_1 and L_2 having q^2+1 lines in common with $\mathcal{M}$. As every three lines of $\mathcal{M}$ span a 5-dimensional subspace which contains q^2+1 lines of $\mathcal{M}$ and which intersects $Q^-(7, q)$ in a nonsingular quadric, there exist at least $((q^4-1)/(q^2-1))-((q^2+1)/2) = (q^2+1)/2$ hyperbolic quadrics $Q_i^+(5, q) \subseteq Q^-(7, q)$, containing L_1, L_2 and q^2-1 other lines of $\mathscr{M}$. Each of them yields two disjoint, conjugate planes π_i and $\bar{\pi}_i$ meeting all lines of the 1-system $\mathscr{M}'_i$, induced by $\mathscr{M}$ in $Q_i^+(5, q)$. By Lemma 4.7, the planes π_i and $\bar{\pi}_i$ are the unique planes in $\langle Q_i^+(5, q^2)\rangle$ meeting all lines of $\mathscr{M}'_i$. As ρ and $\bar{\rho}$ are not contained in any $\langle Q_i^+(5, q^2)\rangle$, this implies that for each $i = 1, 2, ..., (q^2+1)/2$, one of π_i and $\bar{\pi}_i$, say π_i, is contained in ρ and the other one, $\bar{\pi}_i$, is contained in $\bar{\rho}$. The planes π_i share the line through the points $L_1 \cap \rho := l_1$ and $L_2 \cap \rho := l_2$ and the planes $\bar{\pi}_i$ share the line $\overline{l_1 l_2}$. As every two planes π_i and $\bar{\pi}_i$ are polar with respect to the restriction of the polarity of $Q^+(7, q^2)$ to $\langle \pi_i, \bar{\pi}_i \rangle$, each plane $\bar{\pi}_i$ is contained in $l_1^{\perp}$ and $l_2^{\perp}$, with $\perp$ the polarity of $Q^+(7, q^2)$. This implies that $\langle \bar{\pi}_i \mid i = 1, 2, ..., (q^2+1)/2 \rangle$ is contained in both $l_1^{\perp}$ and $l_2^{\perp}$. But the subspace spanned by all planes $\bar{\pi}_i$ is exactly $\bar{\rho}$ and it follows that $\bar{\rho} \subseteq l_1^{\perp} \cap l_2^{\perp}$. As $L_1, L_2 \in \mathscr{M}$ were chosen arbitrarily, this means that

$$\bar{\rho} \subseteq \bigcap_{x \in \mathcal{O}} x^{\perp};$$

thus $\bar{\rho} \subseteq \rho^{\perp}$. Since ρ and $\bar{\rho}$ are 3-dimensional, they must be polar with respect to the polarity of $Q^+(7, q^2)$. This completes the proof of the theorem. ∎

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